

Magneto-inertial convection in rotating fluid spheres

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The onset of convection in the form of magneto-inertial waves in a rotating fluid sphere permeated by a constant axial electric current is studied through a perturbation analysis. Explicit expressions for the dependence of the Rayleigh number on the azimuthal wavenumber are derived in the limit of high thermal diffusivity. Results for the cases of thermally infinitely conducting and of nearly thermally insulating boundaries are obtained.

1. Introduction

Buoyancy driven motions of an electrically conducting fluid in a rotating system and in the presence of a magnetic field represent a fundamental aspect of the dynamics in stars and in planetary interiors. It is thus not surprising that this topic has received much attention ever since Chandrasekhar (1961) and others have presented the first mathematical solutions for problems of this kind. Because of the numerous parameters entering these problems it is not always easy to gain an overview of the physically relevant convection modes. Analytical expressions are thus especially desirable, even though they can usually be obtained only for the onset of convection in planar layers. For some examples of this kind we refer to Chandrasekhar's monograph (1961). From the geo- and astrophysical point of view, magnetoconvection in spherical configurations is of special interest. Here numerical schemes are usually required for the solution of the mathematical problems. For an overview of the subject we refer to the article by Fearn (1994).

In the case of low Prandtl numbers convection typically assumes the form of inertial waves. Buoyancy and viscous dissipation can be treated as perturbations and analytical expressions for the critical values of the Rayleigh numbers for onset of convection can be obtained. This approach has been employed in the case of a rotating sphere without magnetic field first by Zhang (1994). For later analyses with different boundary conditions see Zhang (1995) and Busse & Simitev (2004).

Magneto-inertial modes can be excited in the presence of both, the Coriolis force and the Lorentz force of an imposed magnetic field. Of particular interest from an astrophysical point of view are magneto-inertial waves in rotating fluid spheres with a strong azimuthal magnetic field. Malkus (1967) has demonstrated that for a magnetic field corresponding to an axial electric current with constant density the dispersion relation for magneto-inertial modes can be expressed in terms of the dispersion relation of inertial modes in the absence of a magnetic field. He thus obtained simple analytical expressions for magneto-inertial waves in a configuration of special interest for geophysical and astrophysical applications. In this paper we use the Malkus approach as a starting point

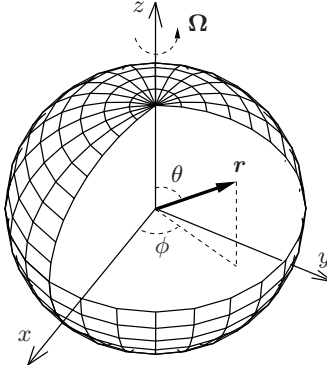


FIGURE 1. Geometrical configuration of the problem. A part of the outer spherical surface is removed to expose the interior of sphere to which the conducting fluid is confined.

for an analysis of the onset of convection in a rotating sphere permeated by a azimuthal magnetic field. In analogy to the treatment of inertial convection in the paper of Busse and Simitev (2004) we investigate the effect of buoyancy and ohmic dissipation as perturbations on magneto-inertial waves. The main result is the determination of expressions for the critical Rayleigh number for the onset of convection as a function of the rotation rate, the strength of the magnetic field and the other parameters of the problem.

In the following we start with the mathematical formulation of the problem in section 2. The special limit of a high ratio of thermal to magnetic diffusivity will be treated in section 3. The general case requires the symbolic evaluation of lengthy analytical expressions and will be presented in section 4. A discussion of the results and an outlook on related problems will be given in the final section 5 of the paper.

2. Mathematical formulation of the problem

We consider a homogeneously heated, self-gravitating fluid sphere as shown in figure 1. The sphere is rotating with a constant angular velocity $\Omega \mathbf{k}$ where \mathbf{k} is the axial unit vector. The fluid is electrically conducting and is characterized by its magnetic diffusivity η , its kinematic viscosity ν and its thermal diffusivity κ . Following Malkus (1967) we assume that the fluid sphere is permeated by a magnetic field of the form $\mathbf{B} = B_0 \mathbf{k} \times \mathbf{r}$ where \mathbf{r} is the position vector with respect to the center of the sphere and r is its length measured in fractions of the radius r_0 of the sphere. Since the Lorentz force like the centrifugal force can be balanced by the pressure gradient a static state exists with the temperature distribution $T_S = T_0 - \beta r_0^2 r^2 / 2$ and the gravity field given by $\mathbf{g} = -\hat{\gamma} r_0 \mathbf{r}$. We use the length r_0 , the time r_0^2 / η , the temperature $\eta^2 / \hat{\gamma} \alpha r_0^4$ and the magnetic flux density $\sqrt{\mu \eta} / r_0$ as scales for the dimensionless description of the problem. The density is assumed to be constant except in the gravity term where its temperature dependence given by $\alpha \equiv (d\rho/dT)/\rho = \text{const.}$ is taken into account. Since we are interested in the onset of convection in the form of small disturbances we can neglect all quadratic terms in the equations. The linearized versions of the equations of motion, the equation of magnetic induction, and the heat equation for the deviation Θ from the static temperature distribution are thus given by

$$\nabla \cdot \tilde{\mathbf{u}} = 0, \quad \nabla \cdot \tilde{\mathbf{b}} = 0, \quad (2.1a)$$

$$\partial_t \tilde{\mathbf{u}} + \tau \mathbf{k} \times \tilde{\mathbf{u}} + \nabla(\pi - \tilde{\mathbf{b}} \cdot \mathbf{j} \times \mathbf{r}) + (\mathbf{j} \times \mathbf{r}) \cdot \nabla \tilde{\mathbf{b}} - \mathbf{j} \times \tilde{\mathbf{b}} = \tilde{\Theta} \mathbf{r} + P_m \nabla^2 \tilde{\mathbf{u}}, \quad (2.1b)$$

$$\partial_t \tilde{\mathbf{b}} - (\mathbf{j} \times \mathbf{r}) \cdot \nabla \tilde{\mathbf{u}} + \mathbf{j} \times \tilde{\mathbf{u}} = \nabla^2 \tilde{\mathbf{b}}, \quad (2.1c)$$

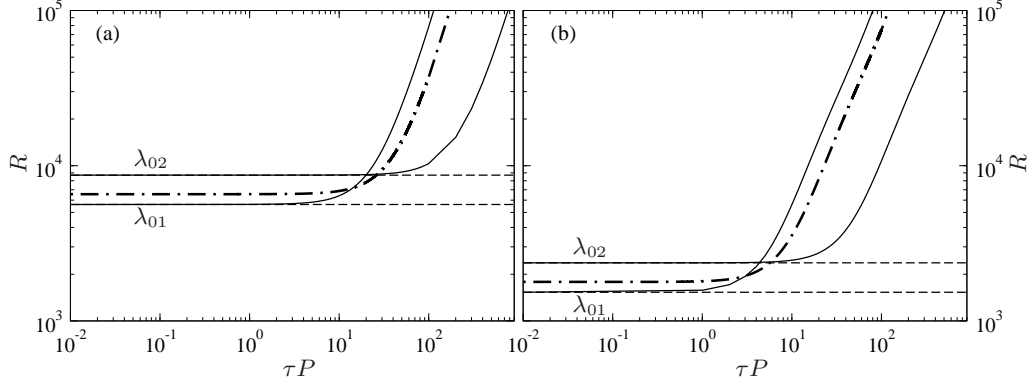


FIGURE 2. Competition of modes with increasing τP in the non-magnetic case discussed in Busse & Simitev (2004). The Rayleigh number R as a function of τP for $m = 0$ (thick dash-dotted lines) and $m = 1$ (thin lines). Results based on the explicit expressions (4.6) and (3.4) from Busse & Simitev (2004) are shown in solid lines and broken lines respectively in the case $m = 1$. (a) Case A, fixed temperature boundary conditions. (b) Case B, insulating thermal boundary conditions.

$$\hat{R} \mathbf{r} \cdot \tilde{\mathbf{u}} + \nabla^2 \tilde{\Theta} - P \partial_t \tilde{\Theta} / P_m = 0, \quad (2.1d)$$

where the dimensionless magnetic field has been assumed in the form $\mathbf{j} \times \mathbf{r} + \tilde{\mathbf{b}}$. Here $2\mathbf{j} = 2j\mathbf{k}$ is the vector of the density of the imposed electric current. The Rayleigh number \hat{R} , the Coriolis parameter τ and the Prandtl number P and the magnetic Prandtl number P_m are defined by

$$\hat{R} = \frac{\alpha \gamma \beta r_0^6}{\eta \kappa}, \quad \tau = \frac{2\Omega r_0^2}{\eta}, \quad P = \frac{\nu}{\kappa}, \quad P_m = \frac{\nu}{\eta}, \quad S = \frac{\eta}{\kappa} = \frac{P}{P_m}. \quad (2.2)$$

We have attached a $\hat{\cdot}$ to the Rayleigh number to remind the reader that the magnetic diffusivity replaces the kinematic viscosity in the definition of \hat{R} . We have also introduced the Strebor number S which is the inverse of what often is called the Roberts number. Without losing generality we may assume an exponential dependence on time t and on the azimuthal angle ϕ . Since both, the velocity field \mathbf{u} and the magnetic field \mathbf{b} , are solenoidal we may use the representation of these vector fields in terms of poloidal and toroidal components,

$$\tilde{\mathbf{u}} = \mathbf{u} \exp(i(\omega \tau t + m\phi)) = (\nabla \times (\nabla v \times \mathbf{r}) + \nabla w \times \mathbf{r}) \exp(i(\omega \tau t + m\phi)), \quad (2.3a)$$

$$\tilde{\mathbf{b}} = \mathbf{b} \exp(i(\omega \tau t + m\phi)) = (\nabla \times (\nabla h \times \mathbf{r}) + \nabla g \times \mathbf{r}) \exp(i(\omega \tau t + m\phi)), \quad (2.3b)$$

$$\tilde{\Theta} = \Theta \exp(i(\omega \tau t + m\phi)). \quad (2.3c)$$

Equation (2.1c) can now be written in the form $i\omega \tau \mathbf{b} = imj\mathbf{u} + \nabla^2 \mathbf{b}$ which allows us to eliminate \mathbf{b} from equation (2.1b),

$$i\omega \left(1 - \frac{m^2}{\omega^2} \gamma^2\right) \mathbf{u} + \left(1 - \frac{m}{\omega} \gamma^2\right) \mathbf{k} \times \mathbf{u} - \nabla^2 \tilde{\pi} = \frac{1}{\tau} \Theta \mathbf{r} + \frac{P_m}{\tau} \nabla^2 \mathbf{u} + \frac{m^2 \gamma^2}{\omega^2 \tau} \nabla^2 \mathbf{u} \\ + \frac{2m\gamma^2}{i\omega^2 \tau} \mathbf{k} \times \nabla^2 \mathbf{u} + \frac{m\gamma}{\omega \tau} \nabla^2 \mathbf{b}_b + \frac{2\gamma}{i\omega \tau} \mathbf{k} \times \nabla^2 \mathbf{b}_b + \frac{P_m}{\tau} \nabla^2 \mathbf{u}_b \quad (2.4)$$

where γ is defined by $\gamma = j/\tau$. In the ∇ -operator the ϕ -derivative is replaced by im , of course. In equation (2.4) the magnetic field \mathbf{b} appears only in the form of the boundary layer correction \mathbf{b}_b which is required since the basic dissipationless solution does not

satisfy all boundary conditions. For the same reason the Ekman layer correction \mathbf{u}_b had to be introduced.

In the following we shall assume the limit of large τ such that in first order of approximation the right hand side of equation (2.4) can be neglected. The left hand side together with the condition $\nabla \cdot \mathbf{u} = 0$ is of the same form as the equation for inertial modes. Among the latter those corresponding to the sectorial spherical harmonics usually lead to the lowest critical Rayleigh numbers for the onset of convection. We shall assume that this property continues to hold as long as γ^2 is sufficiently small. The sectorial inertial modes are given by

$$v_0 = P_m^m(\cos \theta) f(r), \quad w_0 = P_{m+1}^m(\cos \theta) \psi(r), \quad (2.5)$$

with

$$f(r) = r^m - r^{m+2}, \quad \psi(r) = r^{m+1} \frac{2im(m+2)}{(2m+1)(\lambda_0(m^2+3m+2)-m)}, \quad (2.6a)$$

$$\lambda_0 = \frac{1}{m+2} \left(1 \pm \sqrt{\frac{m^2+4m+3}{2m+3}} \right), \quad (2.6b)$$

where $\lambda_0 \tau$ is the frequency of the inertial modes. The sectorial magneto-inertial modes are described by the same velocity field (2.5) and their magnetic field is given by $\mathbf{b}_0 = m\gamma \mathbf{u}_0 / \omega_0$. As before the subscript 0 refers to the dissipationless solution of equations (2.1a,b,c). The frequency ω_0 of the magneto-inertial waves is determined by

$$\lambda_0 = \frac{\omega_0^2 - m^2 \gamma^2}{\omega_0 - m \gamma^2}, \quad (2.7)$$

which yields

$$\omega_0 = \frac{\lambda_0}{2} \pm \sqrt{\frac{\lambda_0^2}{4} + m \gamma^2 (m - \lambda_0)}. \quad (2.8)$$

With account of (2.6b), this dispersion relation allows for a total of four different frequencies ω_0 . For small values of γ^2 these are given by

$$\omega_{01,2} = \frac{1}{m+2} \left(1 \pm \sqrt{\frac{m^2+4m+3}{2m+3}} \right) + m^2 \gamma^2 \left(1 \pm \sqrt{\frac{m^2+4m+3}{2m+3}} \right)^{-1} - m \gamma^2, \quad (2.9a)$$

$$\omega_{03,4} = -m^2 \gamma^2 \left(1 \pm \sqrt{\frac{m^2+4m+3}{2m+3}} \right)^{-1} + m \gamma^2. \quad (2.9b)$$

The upper sign in expression (2.9a) refers to retrogradely propagating modified inertial waves, while the lower sign corresponds to the progradely traveling variety. The effect of the magnetic field tends to increase the absolute value of the frequency in both cases. Expression (2.9b) describes the dispersion of the slow magnetic waves. The upper sign refers to the progradely traveling modified Alfvén waves and the lower sign corresponds to retrogradely propagating modified Alfvén waves.

The magneto-inertial waves described by expressions (2.5) satisfy the condition that the normal component of the velocity field vanishes at the boundary. This property implies that the normal component of the magnetic field vanishes there as well. Additional boundary conditions must be specified when the full dissipative problem described by (2.4) is considered. We shall assume a stress-free boundary with either a fixed temperature

(case A) or a thermally insulating boundary (case B),

$$\mathbf{r} \cdot \mathbf{u} = \mathbf{r} \cdot \nabla(\mathbf{r} \times \mathbf{u})/r^2 = 0 \quad \text{and} \quad \begin{cases} \Theta = 0 & \text{case A} \\ \partial_r \Theta = 0 & \text{case B} \end{cases} \quad \text{at } r = 1. \quad (2.10)$$

Additionally we shall assume an electrically insulating exterior of the sphere which requires

$$g = 0 \quad \text{at } r = 1 \quad (2.11)$$

and the matching of the poloidal magnetic field to a potential field outside the sphere.

Following the procedure of earlier papers (Zhang, 1994; Busse and Simitev, 2004) we use a perturbation approach for solving equations (2.1) in the limit of large τ ,

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1 + \dots, \quad \omega = \omega_0 + \omega_1 + \dots \quad (2.12)$$

After the ansatz (2.12) has been inserted into equation (2.4) such that terms with \mathbf{u}_1 appear on the left hand side, while those with \mathbf{u}_0 and ω_1 appear on the right hand side, we obtain the solvability condition for the equation for \mathbf{u}_1 by multiplying it with \mathbf{u}_0^* and averaging it over the fluid sphere,

$$\begin{aligned} i\omega_1 \langle |\mathbf{u}_0|^2 \rangle & \left(1 + \left(\frac{m^2}{\omega_0^2} - \frac{m(\omega_0^2 - m^2\gamma^2)}{\omega_0^2(\omega_0 - m\gamma^2)} \right) \gamma^2 \right) \\ & = \langle \Theta \mathbf{r} \cdot \mathbf{u}_0^* \rangle + \left(\langle \mathbf{u}_0^* \cdot \nabla^2 \mathbf{u}_0 \rangle \frac{m\gamma}{\omega_0} + \langle \mathbf{u}_0^* \cdot \nabla^2 \mathbf{b}_{0b} \rangle \right) \left(\frac{m}{\omega} - \frac{\omega_0^2 - m^2\gamma^2}{\omega_0^2 - m\omega_0\gamma^2} \right) \frac{\gamma}{\tau}, \end{aligned} \quad (2.13)$$

where the brackets $\langle \dots \rangle$ indicate the average over the fluid sphere and the $*$ indicates the complex conjugate. We have neglected all terms connected with viscous dissipation, i.e. we have assumed the of vanishing P_m , since we wish to focus on the effect of ohmic dissipation. The effects of viscous dissipation have been dealt with in the earlier paper (Busse and Simitev, 2004). Since $\langle \mathbf{u}_0^* \cdot \nabla^2 \mathbf{u}_0 \rangle$ vanishes, as demonstrated in Zhang *et al.* (2001), we must consider only the influence of the boundary layer magnetic field \mathbf{b}_{0b} . It is determined by the equation

$$i\omega_0 \tau \mathbf{b}_{0b} = \nabla^2 \mathbf{b}_{0b}. \quad (2.14)$$

Since the solutions of this equation are characterized by gradients of the order $\sqrt{\tau}$, the boundary layer correction needed for the poloidal component is of the order $\sqrt{\tau}$ smaller than the correction needed for the toroidal component. For large τ we need to take into account only the contribution g_{0b} given by

$$\begin{aligned} g_{0b} & = -g_0(r=1) \exp \left(-(1+is)(1-r)\sqrt{|\omega_0|\tau/2} \right) \\ & = -\frac{m\gamma}{\omega_0} w_0(r=1) \exp \left(-(1+is)(1-r)\sqrt{|\omega_0|\tau/2} \right), \end{aligned} \quad (2.15)$$

where s denotes the sign of ω_0 . The solvability condition thus becomes reduced to

$$\begin{aligned} i\omega_1 \langle |\mathbf{u}_0|^2 \rangle & \left(1 + \left(\frac{m\gamma^2(m-\omega_0)}{\omega_0(\omega_0 - m\gamma^2)} \right) \right) \\ & = \frac{1}{\tau} \langle \Theta \mathbf{r} \cdot \mathbf{u}_0^* \rangle - \frac{3}{2} \frac{m\gamma^2(m-\omega_0)(s+i)}{(\omega_0 - m\gamma^2)\sqrt{2|\omega_0|\tau}} \int_{-1}^1 |P_m^{m+1}|^2 d(\cos \theta) \\ & \quad \times (m+1)(m+2) \left| \frac{2m(m+2)}{(2m+1) \left(\frac{\omega_0^2 - m^2\gamma^2}{\omega_0 - m\gamma^2} (m+1)(m+2) - m \right)} \right|^2, \end{aligned} \quad (2.16)$$

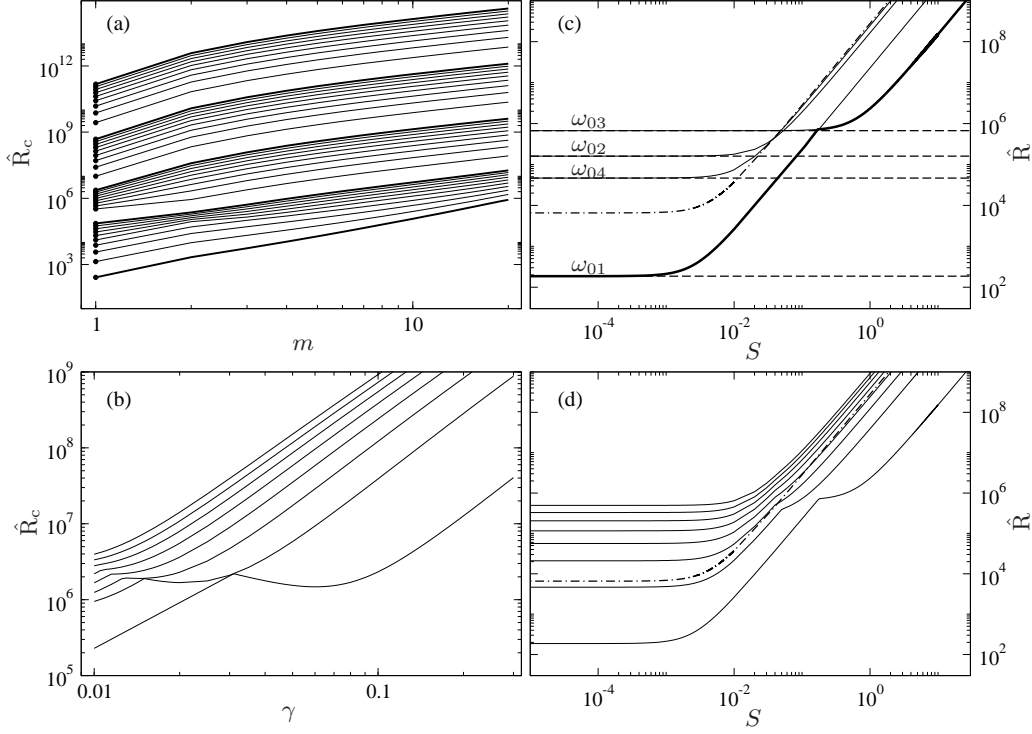


FIGURE 3. Case A. (a) The critical Rayleigh number \hat{R}_c as a function of the wave number m for $\gamma = 0.1$ and $\tau = 10^2 \dots 10^6$ increasing from bottom with log-scale decades given by thick lines. (b) The critical Rayleigh number \hat{R}_c as a function of γ for $S = 1$ and $m = 1 \dots 8$ increasing from bottom. (c) Competition of modes with increasing S for $\gamma = 0.1$ and $m = 1$. Explicit expressions (3.20) in the limit $\tau S \ll 1$ are shown by broken lines. (d) The critical Rayleigh number \hat{R}_c as a function of S for $\gamma = 0.1$ and $m = 1 \dots 8$ increasing from bottom. The axisymmetric mode $m = 0$ is given for comparison in panels (c,d) by a dot-dashed line. In panels (b) to (d) $\tau = 10^4$.

3. Explicit expressions in the limit $\tau S \ll 1$

The equation (2.1d) for Θ can most easily be solved in the limit of vanishing $\omega_0 \tau S$. In this limit we obtain for Θ ,

$$\Theta = P_m^m(\cos \theta) \exp(im\varphi + i\omega\tau t)q(r), \quad (3.17)$$

with

$$q(r) = \hat{R} \left(\frac{m(m+1)r^{m+4}}{(m+5)(m+4) - (m+1)m} - \frac{m(m+1)r^{m+2}}{(m+3)(m+2) - (m+1)m} - cr^m \right), \quad (3.18)$$

where the coefficient c is given by

$$c = \begin{cases} \frac{1}{(m+5)(m+4) - (m+1)m} - \frac{1}{(m+3)(m+2) - (m+1)m} & \text{case A,} \\ \frac{(m+4)/m}{(m+5)(m+4) - (m+1)m} - \frac{(m+2)/m}{(m+3)(m+2) - (m+1)m} & \text{case B.} \end{cases} \quad (3.19)$$

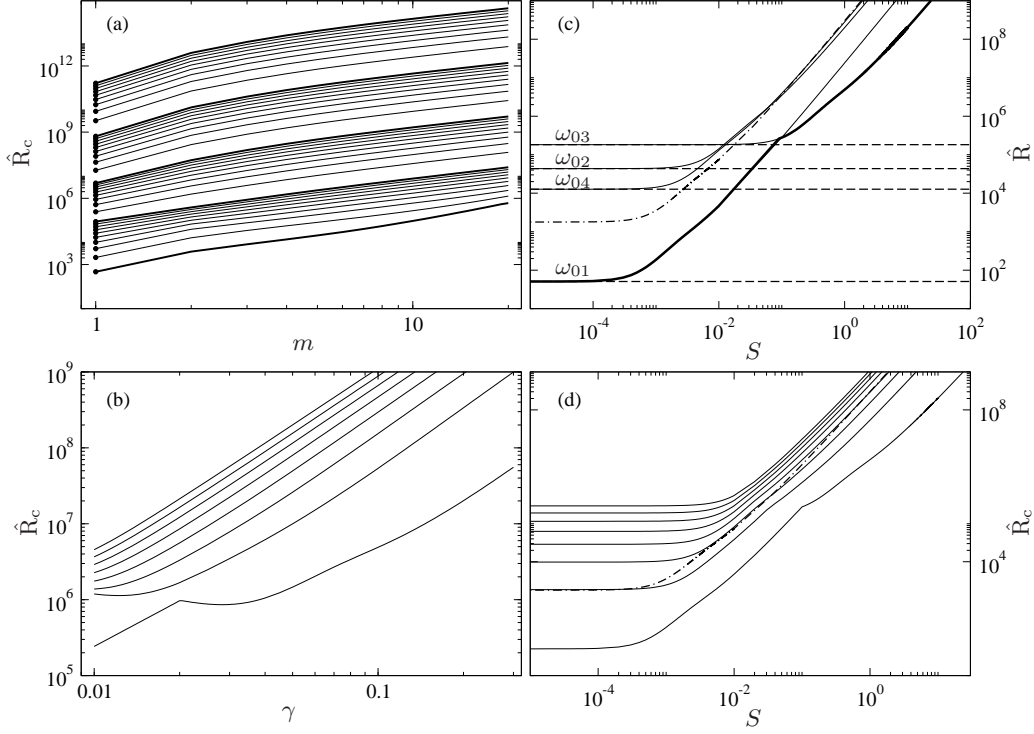


FIGURE 4. Same as figure 3 but in Case B.

Since Θ and the left hand side of equation (2.19) is imaginary, the real parts of the two terms on the right hand side must balance. We thus obtain for R the result

$$\hat{R} = s \sqrt{\frac{\tau}{2|\omega_0|}} \frac{\gamma^2(m - \omega_0)}{(\omega_0 - m\gamma^2)} \left| \frac{m(m+2)}{\frac{\omega_0^2 - m^2\gamma^2}{\omega_0 - m\gamma^2}(m+1)(m+2) - m} \right|^2 \times (2m+9)(2m+7)(2m+5)^2(2m+3) \frac{m+2}{m+1} \frac{1}{b}, \quad (3.20)$$

where the coefficient b assumes the values

$$b = \begin{cases} m(10m+27) & \text{case A,} \\ 14m^2 + 59m + 63 & \text{case B.} \end{cases} \quad (3.21)$$

Obviously the lowest value of \hat{R} is usually reached for $m = 1$, but the fact that there are four different possible values of the frequency ω_0 complicate the determination of the critical value \hat{R}_c . Expression (3.20) is also of interest, however, in the case of spherical fluid shells when the $(m = 1)$ -mode is affected most strongly by the presence of the inner boundary. Convection modes corresponding to higher values of m may then become preferred at onset since their r -dependence decays more rapidly with distance from the outer boundary according to relationships (2.6).

4. Solution of the heat equation in the general case

For the solution of equation (2.1d) in the general case it is convenient to use the Green's function method. The Green's function $G(r, a)$ is obtained as solution of the equation

$$[\partial_r r^2 \partial_r + (-i\omega_0 \tau S r^2 - m(m+1))] G(r, a) = \delta(r - a), \quad (4.22)$$

which can be solved in terms of the spherical Bessel functions $j_m(\mu r)$ and $y_m(\mu r)$,

$$G(r, a) = \begin{cases} G_1(r, a) = A_1 j_m(\mu r) & \text{for } 0 \leq r < a, \\ G_2(r, a) = A j_m(\mu r) + B y_m(\mu r) & \text{for } a < r \leq 1, \end{cases} \quad (4.23)$$

where

$$\mu \equiv \sqrt{-i\omega_0 \tau S}, \quad A_1 = \mu \left(y_m(\mu a) - j_m(\mu a) \frac{y_m(\mu)}{j_m(\mu)} \right), \quad (4.24a, b)$$

$$A = -\mu j_m(\mu a) \frac{y_m(\mu)}{j_m(\mu)}, \quad B = \mu j_m(\mu a). \quad (4.24c, d)$$

A solution of the equation (2.1d) can be obtained in the form

$$q(r) = -m(m+1) \left(\int_0^r G_2(r, a) (a^m - a^{m+2}) a^2 da + \int_r^1 G_1(r, a) (a^m - a^{m+2}) a^2 da \right). \quad (4.25)$$

Evaluations of these integrals for $m = 1$ yield the expressions

$$q(r) = \begin{cases} \frac{2R}{(\omega_0 \tau S)^2} \left(r(\mu^2 + 10) - \mu^2 r^3 - \frac{10(\mu r \cos(\mu r) - \sin(\mu r))}{r^2(\mu \cos \mu - \sin \mu)} \right) & \text{case A,} \\ \frac{2R}{(\omega_0 \tau S)^2} \left(r(\mu^2 + 10) - \mu^2 r^3 - \frac{(\mu^2 - 10)(\mu r \cos(\mu r) - \sin(\mu r))}{r^2(2\mu \cos \mu - (2 - \mu^2) \sin \mu)} \right) & \text{case B.} \end{cases} \quad (4.26)$$

Lengthier expressions are obtained for $m > 1$. Expressions (4.26) can now be used to calculate \hat{R} and ω_1 on the basis of equation (2.16). In the case $m = 1$ we obtain

$$\hat{R} = \frac{189}{20} \frac{s\sqrt{2\tau}\gamma^2(\omega_0 - 1)}{\sqrt{|\omega_0|}(\omega_0 - \gamma^2)(6\lambda_0 - 1)^2} \quad (4.27)$$

$$\times \begin{cases} \left(\mu^{-4} - 525\mu^{-8} - 175 \operatorname{Re} \left\{ \frac{\sin \mu}{\mu^6(\mu \cos \mu - \sin \mu)} \right\} \right)^{-1} & \text{case A,} \\ \left(\mu^{-4} + 231\mu^{-8} + 7 \operatorname{Re} \left\{ \frac{(\mu^5 - 8\mu^3 + 9\mu) \cos \mu - 9 \sin \mu}{\mu^8((\mu^2 - 2) \sin \mu + 2\mu \cos \mu)} \right\} \right)^{-1} & \text{case B,} \end{cases}$$

where $\operatorname{Re}\{\}$ indicates the real part of the term enclosed by $\{\}$. Expressions (4.27) have been plotted as functions of S in figures 3(c) and 4(c) for the cases A and B, respectively. Four curves appear since there are four possible values of ω_0 for each m . For values S of the order 10^{-2} or less, expressions (3.20) are well approached. The retrograde mode corresponding to the positive sign in (2.6b) always yields the lower value of \hat{R} but it loses its preference to the progradely traveling modified Alfvén mode corresponding to the upper sign in (2.9b) as S becomes of the order 10^{-1} or larger. This transition can be understood on the basis of the increasing difference in phase between Θ and u_r with increasing S . While the mode with the largest absolute value of ω is preferred as long as Θ and u_r are in phase, the mode with the minimum absolute value of ω becomes

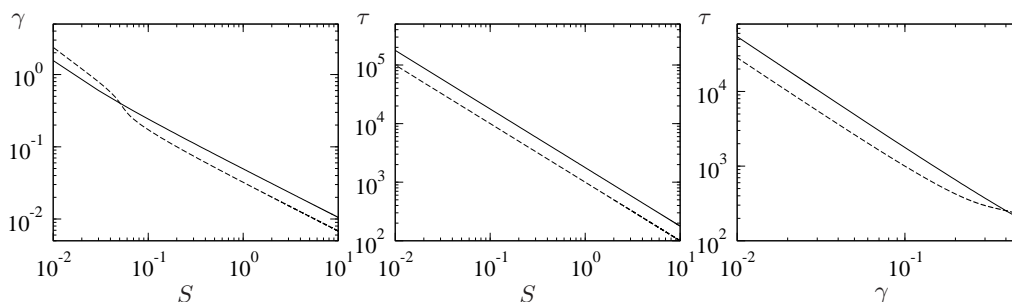


FIGURE 5. The border where the transition from modes characterised by ω_{01} to modes characterised by ω_{03} occurs in various sections of the parameter space. The value of the parameters are $m = 1$, $S = 1$, $\gamma = 0.1$, and $\tau = 5000$ where they are not varied on the axes. Case A is denoted by a solid lines and Case B by broken lines.

preferred as the phase difference increases since the latter is detrimental to the work done by the buoyancy force. The frequency perturbation ω_1 usually makes only a small contribution to ω which tends to decrease the absolute value of ω . This transition shifts towards smaller values of S and γ as τ is increased as illustrated in figure 5. The magneto-inertial convective modes corresponding to higher values of $m = 1 \dots 8$ exhibit similar behaviour as figures 3(d) and 4(d) demonstrate for the cases A and B, respectively. The value $m = 1$ is always the preferred value of the wavenumber, except possibly in a very narrow range near $\gamma = 0.03$ as indicated by figure 3(a,b) in the case A and possibly near $\gamma = 0.02$ in the case B and figure 4(a,b). The axisymmetric mode $m = 0$, given for comparison in panels (c) and (d) in figures 3 and 4, is never preferred in contrast to the purely non-magnetic case where it becomes the critical one near the transition from retrograde to prograde inertial convection modes as seen in figure 2.

For very large values of τ and S the Rayleigh number \hat{R} increases in proportion to $\sqrt{\tau}(\tau S)^2$ for fixed m . In spite of this strong increase Θ remains of the order $\tau^{3/2}S$ on the right hand side of equation (2.1b). The perturbation approach thus continues to be valid for $\tau \rightarrow \infty$ as long as $S \ll 1$ can be assumed. For any fixed low value of S , however, the onset of convection in the form of prograde inertial modes will be replaced with increasing τ at some point by the onset in the form of columnar magneto-convection because the latter obeys an approximate asymptotic relationship for R of the form $\tau^{4/3}$ (see, for example, Eltayeb & Kumar, 1977). This second transition depends on the value of S and will occur at higher values of τ and R for lower values of S . There is little chance that magneto-inertial convection occurs in the Earth's core, for instance, since S is of the order 30000 while the usual estimate for τ is 10^{15} but it might be relevant for understanding of rapidly rotating stars with strong magnetic fields.

5. Discussion

A major result of early studies of magnetoconvection in rapidly rotating systems is the existence of an absolute minimum critical Rayleigh number (Chandrasekhar, 1961, Eltayeb & Kumar, 1977). Imposed magnetic field typically has the effect of increasing the azimuthal scale in comparison with non-magnetic convection. When the magnetic field is weak the small scale of convection rolls leads to a larger value of the critical Rayleigh number as the magnetic field strength decreases. When the magnetic field is strong, it inhibits convection and the critical Rayleigh number increases with the increase of the strength of the imposed field. Consequently, an optimal state of convection exists at intermediate magnetic field strength. In this paper we consider the weak field regime but

a decrease of the critical Rayleigh number with increasing field strength is not observed. This is due to the fact that, non-magnetic inertial convection already has a rather large azimuthal length scale (Busse & Simitev, 2004) which cannot be increased significantly further by the imposed weak magnetic field.

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